Exercise set 5 – Jacobian matrix – Solutions

Exercise 1

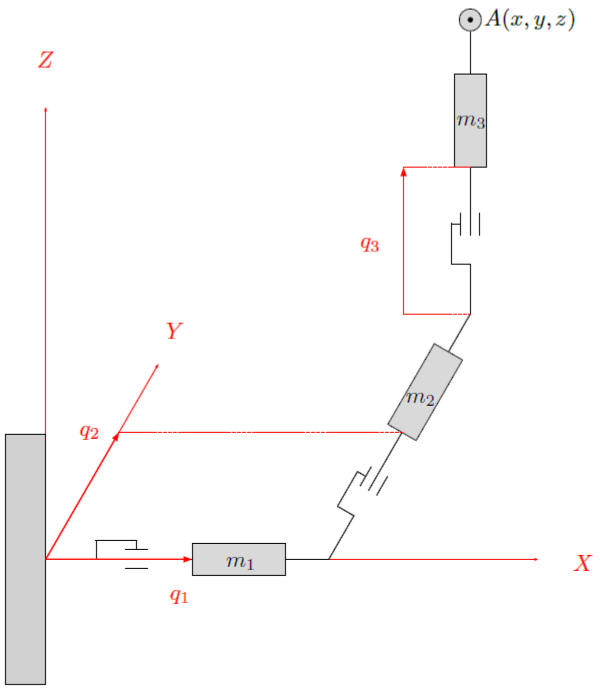
We want to control the axes of a Cartesian machine with 3 DOF in translation. The gravity vector is given by $\mathbf{g} = \begin{pmatrix} 0 \\ 0 \\ -g_0 \end{pmatrix}$ in the robot's base reference frame with $g_0 = 9.8 \mathrm{m/s}^2$.

1. Give the DGM and IGM of this structure.
2. Deduce the Jacobian matrix.
3. By looking at the Jacobian matrix, what can you say about the movements of the tool – are they decoupled? What can you say about the tool position error in the different directions?

Exercise 1 – Solution

We start by drawing a direct coordinate system X, Y, Z in which we represent the machine and its joint movements q_1, q_2, q_3.

The exercise instructions do not give any indication of the reference frame or the orientation of the joint movements, except that the axis Z is vertical and oriented upwards. The following representation is thus chosen:



m_1, m_2 and m_3 are the moving masses associated with each axis.

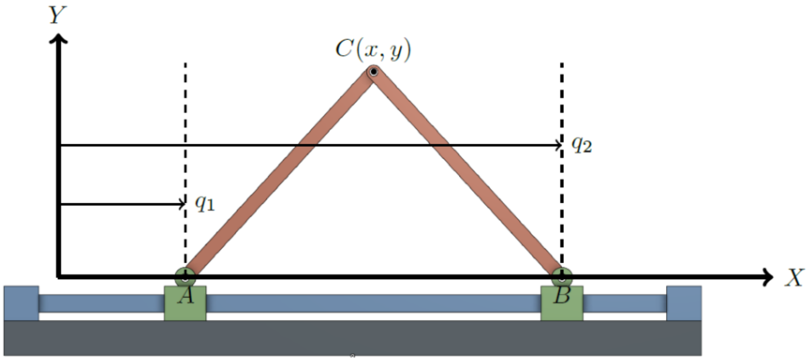
Note that the solution would not be complete without the chosen joint representation.

1. The representation of the DGM is then $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix}$ and that of the IGM is therefore $\begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$.
2. The Jacobian matrix is therefore $\boldsymbol{J} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.
3. By observing the Jacobian matrix, one can see that the movements of the tool of a Cartesian machine are decoupled. The tool position error in all directions is constant and does not depend on the working point of the robot. This is not the case for the next exercise. In addition, for constant joint velocities, the tool

velocities are also constant, the relationship between the two being independent of the working point of the robot.

Exercise 2

Consider the following Lambda robot:



The two arms are of length l ($AC = BC = l$).

Kinematics

1. Is it a parallel or a serial robot?
2. Give the number of DOF.
3. Write down the vectors for:
   (a) the position of the end effector: $c = \begin{pmatrix} \square \\ \square \end{pmatrix}$
   (b) the generalized coordinates: $q = \begin{pmatrix} \square \\ \square \end{pmatrix}$
   (c) the output velocity: $\dot{c} = \begin{pmatrix} \square \\ \square \end{pmatrix}$
   (d) the joint velocity: $\dot{q} = \begin{pmatrix} \square \\ \square \end{pmatrix}$
4. Suggest applications for this robot.

Modeling

5. Find the DGM and IGM of this robot.
6. Deduce the direct and inverse Jacobians of this robot.
7. Explain the utility of Jacobian matrices.
8. Find the singular positions of this robot.

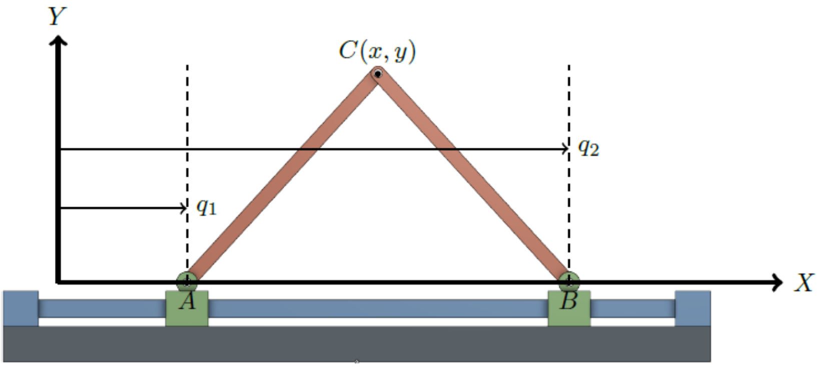
Exercise 2 – Solution

1. This robot has parallel kinematics.
2. It is a robot with 2 DOF, a translation along X and a translation along Y.
   - If both motorized joints move in the same direction at the same speed, the movement is only along X.
   - If both motorized joints move in opposite directions at the same speed, the movement is only along Y.
   - In all other cases, the movement is coupled.
3. (a) The tool position vector is $\mathbf{c} = \begin{pmatrix} x \\ y \end{pmatrix}$.

(b) The generalized coordinate vector is $\mathbf{q} = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$.

(c) The output velocity vector is $\dot{\mathbf{c}} = \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}$.

(d) The joint velocity vector is $\dot{\mathbf{q}} = \begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \end{pmatrix}$.



4. The applications of this robot are multiple:
   - Seated position rehabilitation movements, figure Lambda (a), https://www.lhs-sa.ch/
   - Gait trainer from Reha Technology, figure Lambda (b), https://www.rehatechnology.com/en

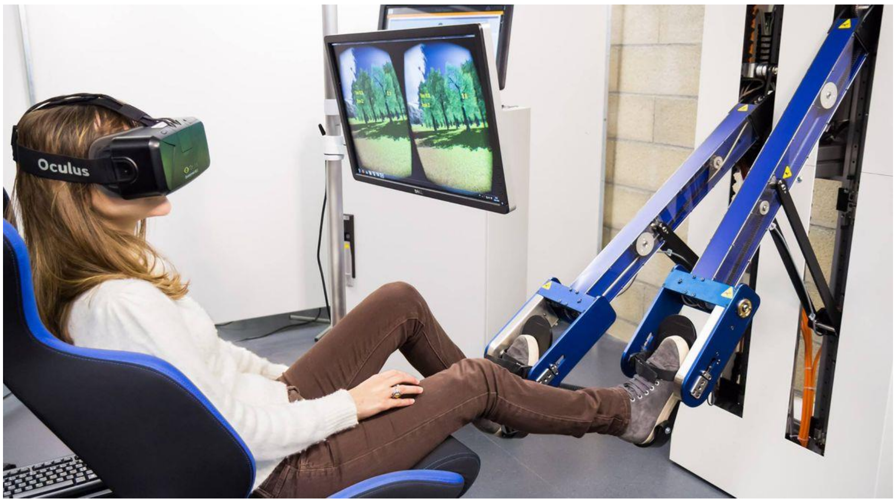


Fig. Lambda (a) LHS device (from LHS-SA) b) R-Geo1 device

5. Direct geometric model:

$$\begin{cases} x &= q_1 + \dfrac{q_2 - q_1}{2} = \dfrac{q_1}{2} + \dfrac{q_2}{2} \\ y &= \sqrt{l^2 - \left(\dfrac{q_2 - q_1}{2}\right)^2} \end{cases}$$

Inverse geometric model:

$$\begin{cases} q_1 &= x - \sqrt{l^2 - y^2} \\ q_2 &= x + \sqrt{l^2 - y^2} \end{cases}$$

6. Direct Jacobian matrix: $\mathbf{J}$ is the derivative of the DGM with respect to the joint variables, namely:

$$\mathbf{J} = \begin{pmatrix} \frac{\partial x}{\partial q_1} & \frac{\partial x}{\partial q_2} \\ \frac{\partial y}{\partial q_1} & \frac{\partial y}{\partial q_2} \end{pmatrix}$$
$$= \begin{pmatrix} 1/2 & 1/2 \\ \frac{q_2 - q_1}{4\sqrt{l^2 - \left(\frac{q_2 - q_1}{2}\right)^2}} & \frac{q_1 - q_2}{4\sqrt{l^2 - \left(\frac{q_2 - q_1}{2}\right)^2}} \end{pmatrix}$$

Inverse Jacobian matrix: $\mathbf{J}^{-1}$ is the inverse matrix of the direct Jacobian matrix $\mathbf{J}$, and also the derivative of the IGM with respect to the tool coordinates:

$$\mathbf{J}^{-1} = \begin{pmatrix} \frac{\partial q_1}{\partial x} & \frac{\partial q_1}{\partial y} \\ \frac{\partial q_2}{\partial x} & \frac{\partial q_2}{\partial y} \end{pmatrix}$$
$$= \begin{pmatrix} 1 & \frac{y}{\sqrt{l^2 - y^2}} \\ 1 & \frac{-y}{\sqrt{l^2 - y^2}} \end{pmatrix}$$

7. The Jacobian matrices have multiple applications:

   (a) Relation between the speeds:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \mathbf{J}(q_1, q_2) \begin{pmatrix} \dot{q_1} \\ \dot{q_2} \end{pmatrix} = f_{\text{direct}}(q_1, q_2, \dot{q_1}, \dot{q_2})$$

   From the knowledge of the speeds of motors 1 and 2, we can deduce the speeds at the level of the tool. This matrix is a relationship between the tool and joint speeds: the direct Jacobian matrix is also a reduction matrix.

$$\begin{pmatrix} \dot{q_1} \\ \dot{q_2} \end{pmatrix} = \mathbf{J}^{-1}(x, y) \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = f_{\text{inverse}}(x, y, \dot{x}, \dot{y})$$

   Thanks to the knowledge of the desired speeds at the tool level, we can choose the motors (it is a question of dimensioning only the speed of the motors). It should be noted that these joint speeds (motors) depend on the working position of the robot, and it is necessary to do an in-depth analysis to size the worst case.

   (b) Relation between joint and tool differential movements is as follows:

$$\begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} = \mathbf{J}(q_1, q_2) \begin{pmatrix} \Delta q_1 \\ \Delta q_2 \end{pmatrix} = f_{\text{direct}}(q_1, q_2, \Delta q_1, \Delta q_2)$$

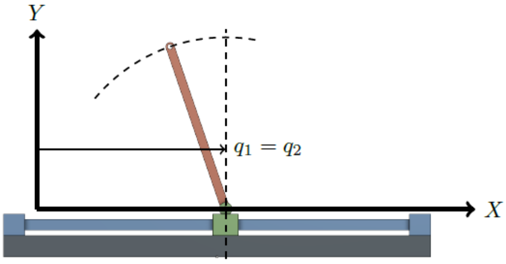
   Thus, from the joint resolutions (sensors at the level of the motors), we deduce the resolutions at the level of the tool. Note that this tool resolution depends on the working position.

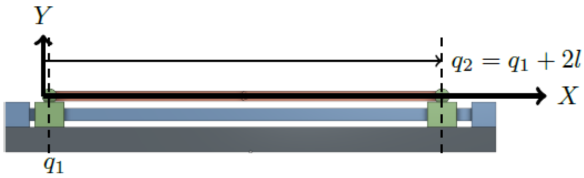
$$\begin{pmatrix} \Delta q_1 \\ \Delta q_2 \end{pmatrix} = \mathbf{J}^{-1}(x, y) \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} = f_{\text{inverse}}(x, y, \Delta x, \Delta y)$$

   From the knowledge of the tool resolution specified by the specifications (Δx and Δy), we deduce the joint resolutions at the level of the motors (Δq_1 and Δq_2). These joint resolutions (motor sensors) depend on the working position of the robot, and it is necessary to do an in-depth analysis to size the worst case.

8. The direct Jacobian matrix is: $\mathbf{J} = \begin{pmatrix} \dfrac{\dfrac{1/2}{q_2 - q_1}}{4\sqrt{l^2 - \left(\dfrac{q_2 - q_1}{2}\right)^2}} & \dfrac{\dfrac{1/2}{q_1 - q_2}}{4\sqrt{l^2 - \left(\dfrac{q_2 - q_1}{2}\right)^2}} \end{pmatrix}$

and its determinant is $\dfrac{q_1 - q_2}{4\sqrt{l^2 - \left(\dfrac{q_2 - q_1}{2}\right)^2}}$. The singularities are found for a null determinant and for invalid values, i.e. $q_1 = q_2$ and $q_2 = 2l + q_1$





a) Parallel singularity: $q_1 = q_2$

b) Serial singularity: $q_2 = 2l + q_1$

We could have also found these values by using a drawing, as in the figure above.

Exercise 3

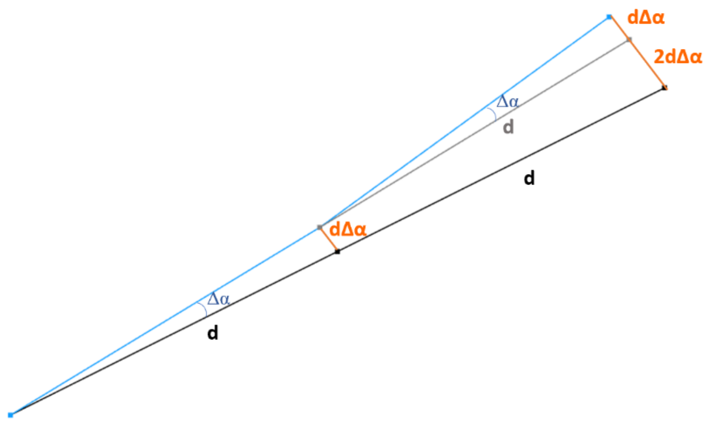
The objective of this exercise is to calculate the positioning resolution projected from angular motion.

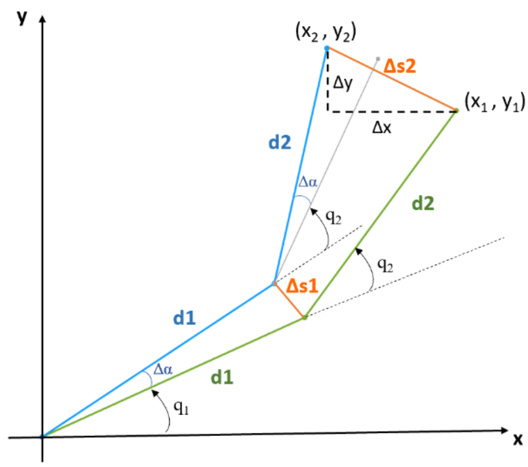
Consider a robot with RR kinematics. The joint resolution of each axis is $\Delta\alpha = 0.001°$, and the length of each segment is $d = 350\text{mm}$.

1. Give the resolution (in units of length) Δl_i at the end of each segment.
2. Is this resolution constant (at the end of the second segment)? Explain.
3. Explain how the Jacobian matrix is useful for sizing the motors of this robot.

Exercise 3 - Solution

1.





At the end of the first arm, the position resolution is **$\Delta s_1 = d\Delta\alpha \approx 6$ µm**. (Note that $\Delta\alpha$ should be in radians.)

For an extended configuration as in the left Figure above, at the end of the second arm, the resolution is at least double (**$2d\Delta\alpha$**), to which we have to add the contribution of the resolution of the second arm **$d\Delta\alpha$**. This therefore reaches **$\Delta s_2 = 3d\Delta\alpha \approx 18$ µm**.

For other configurations (see the right Figure above), one can use the Jacobian matrix to calculate the resolution. As derived in the lecture, we can write:

$$\begin{pmatrix}\Delta x\\ \Delta y\end{pmatrix} = \begin{pmatrix}-d_1\sin(q_1) - d_2\sin(q_1+q_2) & -d_2\sin(q_1+q_2)\\ d_1\cos(q_1) + d_2\cos(q_1+q_2) & d_2\cos(q_1+q_2)\end{pmatrix}\begin{pmatrix}\Delta\alpha_1\\ \Delta\alpha_2\end{pmatrix}$$

With $d_1 = d_2 = d$ and $\Delta\alpha_1 = \Delta\alpha_2 = \Delta\alpha$, we have:

$$\begin{pmatrix}\Delta x\\ \Delta y\end{pmatrix} = \begin{pmatrix}-d\sin(q_1) - d\sin(q_1+q_2) & -d\sin(q_1+q_2)\\ d\cos(q_1) + d\cos(q_1+q_2) & d\cos(q_1+q_2)\end{pmatrix}\begin{pmatrix}\Delta\alpha\\ \Delta\alpha\end{pmatrix}$$

$$\Rightarrow \begin{cases}\Delta x &= -d\Delta\alpha\left(\sin(q_1) + \sin(q_1+q_2) + \sin(q_1+q_2)\right)\\ \Delta y &= d\Delta\alpha\left(\cos(q_1) + \cos(q_1+q_2) + \cos(q_1+q_2)\right)\end{cases}$$

$$\Rightarrow \begin{cases}\Delta x = -d\Delta\alpha(\sin(q_1) + 2\sin(q_1+q_2))\\ \Delta y = d\Delta\alpha(\cos(q_1) + 2\cos(q_1+q_2))\end{cases}$$

$$\begin{aligned}\Delta s_2 &= \sqrt{\Delta x^2 + \Delta y^2}\\ &= d\Delta\alpha\sqrt{(\sin(q_1) + 2\sin(q_1+q_2))^2 + (\cos(q_1) + 2\cos(q_1+q_2))^2}\\ &= d\Delta\alpha\sqrt{5 + 4\sin(q_1)\sin(q_1+q_2) + 4\cos(q_1)\cos(q_1+q_2)}\\ &= d\Delta\alpha\sqrt{5 + 4\cos(q_2)}\end{aligned}$$

$^*\cos(q_2) = \cos(q_1 + q_2 - q_1) = \cos(q_1+q_2)\cos(-q_1) - \sin(q_1+q_2)\sin(-q_1) = \cos(q_1+q_2)\cos(q_1) + \sin(q_1+q_2)\sin(q_1)$

Given the upper and lower bounds of cosine (1 and -1), we can determine the upper and lower bounds of Δs_2:

$$d\Delta\alpha \le \Delta s_2 \le 3d\Delta\alpha$$

$$6\ \mu\text{m} \le \Delta s_2 \le 18\ \mu\text{m}$$

This means that the smallest displacement for a quantization error of 0.001 degree is 6 µm.

2. The end effector resolution is not constant. It depends on the position of the robot. The expression of the Jacobian matrix seen just above defines this variation:

$$\begin{pmatrix}\Delta x\\ \Delta y\end{pmatrix} = \begin{pmatrix}-d\sin(q_1) - d\sin(q_1+q_2) & -d\sin(q_1+q_2)\\ d\cos(q_1) + d\cos(q_1+q_2) & d\cos(q_1+q_2)\end{pmatrix}\begin{pmatrix}\Delta\alpha\\ \Delta\alpha\end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix}\Delta x\\ \Delta y\end{pmatrix} = \mathbf{J}(q_1, q_2)\begin{pmatrix}\Delta\alpha\\ \Delta\alpha\end{pmatrix}$$

This expression makes it possible to calculate, for each position, the variation of the resolution.

A simulation of $\Delta x, \Delta y$ and $\Delta s_2 = \sqrt{\Delta x^2 + \Delta y^2}$, as functions of q_1 and q_2 (q_1 varying from 0 to 2π and q_2 varying from $-\pi$ to π) is provided on Moodle.
(ref. python script uploaded on Moodle)

3. We know that the speeds of the tool are linked to the joint speeds by the Jacobian matrix as follows:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \mathbf{J}(q_1, q_2) \begin{pmatrix} \dot{q_1} \\ \dot{q_2} \end{pmatrix}$$
$$\Leftrightarrow \begin{pmatrix} \dot{q_1} \\ \dot{q_2} \end{pmatrix} = \mathbf{J}^{-1}(q_1, q_2) \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}$$

   Therefore, for given trajectory specifications (requirement in terms of velocities of the end effector), we can calculate the requirements for the actuators and therefore choose properly the motors.